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FINITE AMPLITUDE AND FREE VIBRATIONS OF A BODY SUPPORTED BY INCOMPRESSIBLE, NONLINEAR VISCOELASTIC SHEAR MOUNTINGS

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Abstract—A constitutive equation for an incompressible, isotropic, nonlinear viscoelastic solid of differential type, a class which includes the Voigt–Kelvin solid of classical linear viscoelasticity, is applied to study the quasi-static response of the material in a simple shearing deformation superimposed on a given static homogeneous strain. The Cauchy stress is determined and general relations that characterize creep and recovery phenomena are obtained. Specific equations are derived for a viscoelastic Mooney–Rivlin model. Then the finite amplitude, damped, free vibration of a rigid body supported symmetrically by viscoelastic Mooney–Rivlin shear mountings is examined, and solutions are given for heavily damped and lightly damped motions. The effects of the primary static deformation on creep and recovery phenomena of the shear blocks, and its effects on the frequency, damping, and logarithmic decrement characteristic of the motion are described analytically and illustrated graphically. Effects of the ultimate equilibrium shear induced by the load also are described. Universal frequency and damping relations for viscoelastic Mooney–Rivlin and neo-Hookean models are noted. It is shown that the primary homogeneous deformation plays an important role in determination of all aspects of the mechanical response. General equations for the exact solution of the problem for free vibrations of a load on nonlinear, perfectly elastic shear mountings also are provided.

1. INTRODUCTION

Engineering applications of shear mountings are well-known. Specific examples, including the effects of shear in biological members, have been described in several recent papers by Beatty (1984, 1988, 1989a), Beatty and Bhattacharyya (1989), and Bhattacharyya (1990). These studies have yielded a variety of physical results which have shown that a simple shear model provides significant mathematical simplicity in the study of finite amplitude vibrations of a load supported by rubber shear springs. Various, rather general situations have been investigated.

The undamped, large amplitude, periodic free vibration of a load supported symmetrically by arbitrary isotropic elastic shear mounts has been studied by Beatty (1988). It is assumed only that the shear response function is a positive, even function of the amount of shear. When the shear response function is a constant, it is found that the finite motion of the load is always simple harmonic. The Mooney–Rivlin, Hadamard, and Blatz–Ko models are examples for which this result holds. Otherwise, the frequency is amplitude-dependent. This was illustrated exactly in application to a class of hyperelastic biological tissues. An approximate frequency/amplitude relation was obtained for a soft tissue whose shear response is a quadratic function of the amount of shear. However, for the same problem, the exact solution in terms of elliptic functions may be read from the general result derived earlier for a certain class of rubber-like quadratic materials (Beatty, 1984). This class includes the aforementioned special models having a constant or a quadratic shear response function in a simple shear deformation. The general model explored by Beatty (1988) was subsequently applied to study the stability of the oscillatory motion of a load attached by simple shear mountings to a steadily rotating vehicle (Beatty, 1989a). General conditions for stability are described in simple physical terms; but study of the

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nature of the mechanical response in special cases would depend upon the particular form of the shear response. The rich variety of results possible in specific cases was then illustrated for the class of quadratic materials mentioned previously (Beatty, 1984). Free vibrational motion is studied by Beatty and Bhattacharyya (1989); forced motion is examined by Bhattacharyya (1989).

In all of these studies, the shear springs are considered to be ideally elastic rubber-like materials, and the springs are undeformed prior to shearing. Although it is well-known that rubber exhibits viscoelastic behavior, the damping effect typical of rubber springs previously was ignored for simplicity. Thus, our purpose in this work is to consider the effects of viscous damping on the finite amplitude oscillations of a load supported symmetrically between two simple shear blocks. In this study, the blocks also may be homogeneously deformed prior to shearing.

A constitutive equation for a general class of incompressible, isotropic viscoelastic materials of differential type will be described in Section 2. We then consider a special class for which the Cauchy stress is at most a linear function of the stretching tensor, a class that includes the well-known Voigt–Kelvin solid of linear viscoelasticity theory. This otherwise nonlinear constitutive equation predicts creep and recovery phenomena typical of linear viscoelastic materials, but here extended to include familiar kinds of incompressible, hyperelastic materials with linear viscosity. We name these materials viscohyperelastic materials, and exhibit particular kinds identified as Mooney–Rivlin and neo-Hookean varieties, the latter being a special case of the former.

The nonlinear theory is applied in Section 3 to study the quasi-static response of the material in a simple shearing deformation, superimposed on a given static, finite homogeneous strain. The Cauchy stress is determined. In Section 4, general relations that characterize creep and recovery phenomena are obtained and specific results are derived for a viscoelastic Mooney–Rivlin material. Then, in Section 5, solutions for the finite amplitude, heavily damped and lightly damped, free vibrations of a rigid body supported symmetrically by viscoelastic Mooney–Rivlin shear mountings are described. The effects of the primary homogeneous strain on creep and recovery phenomena of the material, and its effects on the frequency, the damping, and the logarithmic decrement typical of the physical response are described analytically and illustrated graphically. In addition, effects of the ultimate equilibrium shear are discussed. Universal frequency and damping relations which are independent of the elastic or viscous material parameters are obtained for our viscoelastic Mooney–Rivlin and neo-Hookean materials. The analysis shows that the primary homogeneous deformation plays a significant role in the determination of all aspects of the mechanical response of the shearing oscillator. General equations for the exact solution of the problem for free vibrations of a load on nonlinear, perfectly elastic shear mountings are given at the end.

2. THE CONSTITUTIVE EQUATION FOR A NONLINEAR VISCOELASTIC SOLID

The constitutive equation for an incompressible, isotropic, nonlinear viscoelastic solid of differential type will be introduced, and afterwards two special hyperelastic varieties will be identified. It will be shown that this model is a generalized form of the well-known Voigt–Kelvin solid of classical linear viscoelasticity theory.

To begin, we shall need to recall the Cauchy–Green deformation tensor $\mathbf{B}$ and the spatial velocity gradient tensor $\mathbf{L} \equiv \operatorname{grad} \mathbf{v}(\mathbf{x}, t)$ (Truesdell and Noll, 1965). These are defined in terms of the deformation gradient tensor $\mathbf{F}$ in accordance with

$$\mathbf{F} \equiv \frac{\partial \mathbf{x}(\mathbf{X}, t)}{\partial \mathbf{X}}, \quad \mathbf{B} \equiv \mathbf{F}\mathbf{F}^T, \quad \mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}, \tag{1}$$

where $\mathbf{x}(\mathbf{X}, t)$ is the current position vector of the particle initially at the place $\mathbf{X}$ in a fixed frame φ. A superimposed dot denotes the usual material time derivative in φ. We also recall the stretching tensor $\mathbf{D} \equiv \frac{1}{2}(\mathbf{L}+\mathbf{L}^T)$. When the material is incompressible, the following constraints on the principal invariants of **B** and **D** must be respected for all motions of the body:

$$I_3(\mathbf{B}) = \det \mathbf{B} = 1, \quad I_1(\mathbf{D}) = \operatorname{tr} \mathbf{D} = \operatorname{div} \mathbf{v} = 0. \tag{2}$$

Now let us consider a class of isotropic, incompressible viscoelastic materials of differential type for which the extra Cauchy stress $\mathbf{T}_E = \mathbf{T}+p\mathbf{1}$ is an isotropic function $\mathscr{J}$ of **B** and **D** so that

$$\mathbf{T} \equiv -p\mathbf{1}+\mathscr{J}(\mathbf{B},\mathbf{D}), \tag{3}$$

where p is an undetermined pressure. We note that the model (3) is a member of the general class of materials of the differential type described by Truesdell and Noll (1965). It is also known as a Rivlin–Ericksen material of grade 1. Accordingly, the isotropic function $\mathscr{J}$ has the Rivlin–Ericksen representation

$$\mathscr{J} = \phi_1\mathbf{B}+\phi_2\mathbf{D}+\phi_3\mathbf{B}^2+\phi_4\mathbf{D}^2+\phi_5(\mathbf{BD}+\mathbf{DB})+\phi_6(\mathbf{B}^2\mathbf{D}+\mathbf{DB}^2)$$
$$+\phi_7(\mathbf{BD}^2+\mathbf{D}^2\mathbf{B})+\phi_8(\mathbf{B}^2\mathbf{D}^2+\mathbf{D}^2\mathbf{B}^2), \tag{4}$$

where ϕ_i $(i = 1,2,\ldots,8)$ are certain scalar-valued, isotropic functions of **B** and **D** (Rivlin and Ericksen, 1955). See also Truesdell and Noll (1965).

The Cayley–Hamilton theorem may be used to recast (4) in terms of $\mathbf{B}^{-1}$. Thus, we wish to direct attention to a particular subclass of these incompressible viscoelastic materials whose constitutive equation (3) is given by

$$\mathbf{T} = -p\mathbf{1}+\beta_1\mathbf{B}+\beta_{-1}\mathbf{B}^{-1}+2\eta\mathbf{D}. \tag{5}$$

The constant η is the viscosity coefficient and β_1 and β_{-1} are scalar-valued functions of the principal invariants I_1 and I_2 of **B** alone. When $\eta = 0$, (5) yields the familiar constitutive equation for an incompressible, isotropic elastic solid. Thus, (5) describes the uncoupled linear viscous and nonlinear elastic response of an isotropic, incompressible material, a possible special case among nonlinear theories with linear viscosity described in Truesdell and Noll (1965, p. 114). See also Narain (1986).

2.1. *Relation to linearized viscoelasticity theory*

The linearized infinitesimal theory related to (5) may be easily derived. We let $\mathbf{F} = \mathbf{1}+\mathbf{G}$, where **G** is the usual infinitesimal deformation gradient from the natural state, and recall that the infinitesimal engineering strain $\boldsymbol{\varepsilon} = \frac{1}{2}(\mathbf{G}+\mathbf{G}^T)$. Then upon neglecting all products of **G** and $\dot{\mathbf{G}}$, we find by (1) that

$$\mathbf{B} = \mathbf{1}+2\boldsymbol{\varepsilon}, \quad \mathbf{D} = \dot{\boldsymbol{\varepsilon}}, \quad \operatorname{tr}\boldsymbol{\varepsilon} = 0, \quad \operatorname{tr}\dot{\boldsymbol{\varepsilon}} = 0. \tag{6}$$

The last two relations are the first order approximations to the incompressibility constraints in (2). Thus, to the first order in the infinitesimal strain $\boldsymbol{\varepsilon}$ and strain rate $\dot{\boldsymbol{\varepsilon}}$, the constitutive equation (5) is approximated by

$$\mathbf{T} = -\hat{p}\mathbf{1}+2G\boldsymbol{\varepsilon}+2\eta\dot{\boldsymbol{\varepsilon}}. \tag{7}$$

In (7), $\hat{p}$ is another arbitrary, undetermined hydrostatic pressure, $G \equiv \beta_1(3,3)-\beta_{-1}(3,3)$ denotes the constant shear modulus of the natural state, and **T** is now the same as the engineering stress tensor. We recognize (7) as the constitutive equation for the familiar incompressible Voigt–Kelvin solid. Therefore, (5) describes one kind of generalized incompressible Voigt–Kelvin material for finite deformations. It is evident from (4), however, that other generalized varieties of linearly viscous, nonlinearly elastic materials exist which will reduce to the same linearized equation (7).

2.2. *Nonlinear incompressible viscohyperelastic materials*

When the elastic response functions may be characterized by a strain energy density $\Sigma(I_1, I_2)$, per unit volume, so that

$$\beta_1 = 2\frac{\partial\Sigma}{\partial I_1}, \quad \beta_{-1} = -2\frac{\partial\Sigma}{\partial I_2}, \tag{8}$$

the model (5) is called an incompressible, *viscohyperelastic material.* A viscoelastic Mooney–Rivlin material is an example for which the elastic response functions (8) are constants. That is, the strain energy is a linear function of the first and the second invariants of $\mathbf{B}$ (Beatty, 1987). We thus write

$$\Sigma = \frac{G}{2(1+\alpha)}[(I_1-3)+\alpha(I_2-3)], \tag{9}$$

where G is the constant elastic shear modulus and $\alpha > 0$ is another material constant, usually between 0 and 1. Hence, the constant elastic response functions (8) are expressed by

$$\beta_1 = \frac{G}{1+\alpha}, \quad \beta_{-1} = -\frac{\alpha G}{1+\alpha}, \tag{10}$$

and the constitutive equation for our viscoelastic Mooney–Rivlin material is given by

$$\mathbf{T} = -p\mathbf{1} + \frac{G}{1+\alpha}[\mathbf{B} - \alpha\mathbf{B}^{-1}] + 2\eta\mathbf{D}. \tag{11}$$

The special case $\alpha = 0$ defines the viscoelastic neo-Hookean model:

$$\mathbf{T} = -p\mathbf{1} + G\mathbf{B} + 2\eta\mathbf{D}. \tag{12}$$

Thus, the viscoelastic neo-Hookean material is very similar to the linearized form in (7). For brevity, we sometimes refer to (11) and (12) as the Mooney–Rivlin and neo-Hookean models, respectively. It is clear that other kinds of models may be introduced.

A compressible class of viscoelastic materials may be defined similarly upon replacement in (5) of $-p$ by another elastic response function β_0, for example. In this case, the three response functions will depend on all three of the principal invariants of $\mathbf{B}$ alone. It should be noted also that for compressible materials an additional term $\phi_0\mathbf{1}$ must be appended in (4). We shall return to this topic in a separate paper. An easy application of the general theory for incompressible materials will be studied next.

3. SIMPLE SHEAR SUPERIMPOSED ON A FINITE TRIAXIAL STRETCH

We now consider a rigid body of mass M on a smooth surface inclined at an angle θ with the horizontal plane and supported symmetrically between identical viscoelastic rubber springs of original length L and cross-sectional area A. We shall suppose that by application of surface tractions alone each spring is initially subjected the same homogeneous, quasi-static triaxial deformation leading to an ultimate equilibrium configuration with coordinate stretches λ_k so that $\lambda_1\lambda_2\lambda_3 = 1$. We shall refer to this ultimate equilibrium configuration of homogeneous strain as the *homogeneous*, or *predeformed state*. The springs are then bonded

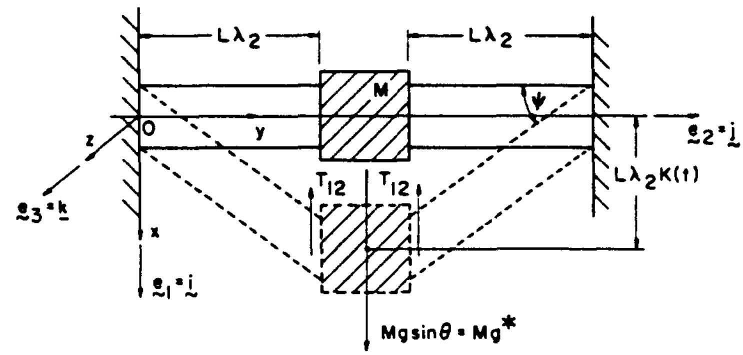

Fig. 1. A rigid load M supported symmetrically between identical viscoelastic rubber shear springs subjected initially to a homogeneous deformation with coordinate stretches λ_k.

to the body at one end and to rigid supports at the other, as suggested in Fig. 1. The equilibrium stress distribution in the homogeneous state will be indicated later; other details are similar to those provided by Beatty and Zhou (1990). Since the shear mountings are identical, however, the springs exert no resultant force on the load M in the homogeneous state.

We now suppose that when the load is released to slide on the inclined surface, each block executes a further time-dependent simple shearing deformation of amount $K(t)$. Hence, the shearing motion is defined by the following rectangular Cartesian coordinate relation for the present place (x, y, z) occupied by the particle whose place was at (X, Y, Z) in the natural, undeformed state:

$$x = \lambda_1 X + K(t)\lambda_2 Y, \quad y = \lambda_2 Y, \quad z = \lambda_3 Z. \tag{13}$$

Of course, the superimposed simple shear (13) is an ideal deformation. For simplicity, we have ignored bending and other evident end effects which usually will accompany the shearing. With $K(0) = 0$, (13) describes the primary triaxial deformation described above. We note that for a time-independent shear, (13) coincides with the example studied by Wineman and Gandhi (1984), Rajagopal and Wineman (1987), and Beatty (1989b).

Let $\{\mathbf{e}_k\} = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ denote the usual rectangular Cartesian basis in the directions of x, y and z, respectively, as shown in Fig. 1. Then $\mathbf{e}_{jk} = \mathbf{e}_j \otimes \mathbf{e}_k$ defines the associated Cartesian tensor product basis. Use of (13) in (1) yields

$$\mathbf{F} = \lambda_1 \mathbf{e}_{11} + \lambda_2 \mathbf{e}_{22} + \lambda_3 \mathbf{e}_{33} + K\lambda_2 \mathbf{e}_{12}, \tag{14}$$

$$\mathbf{B} = (\lambda_1^2 + K^2\lambda_2^2)\mathbf{e}_{11} + \lambda_2^2 \mathbf{e}_{22} + \lambda_3^2 \mathbf{e}_{33} + K\lambda_2^2(\mathbf{e}_{12} + \mathbf{e}_{21}), \tag{15}$$

$$\mathbf{B}^{-1} = \lambda_1^{-2}\mathbf{e}_{11} + \lambda_3^{-2}\mathbf{e}_{33} + (\lambda_2^{-2} + K^2\lambda_1^{-2})\mathbf{e}_{22} - K\lambda_1^{-2}(\mathbf{e}_{12} + \mathbf{e}_{21}). \tag{16}$$

$$\mathbf{D} = \tfrac{1}{2}\dot{K}(\mathbf{e}_{12} + \mathbf{e}_{21}). \tag{17}$$

The relevant principal invariants are then found to be

$$\begin{aligned} I_1(\mathbf{B}) &= \lambda_1^2 + \lambda_3^2 + \lambda_2^2(K^2 + 1), \\ I_2(\mathbf{B}) &= \lambda_2^{-2} + \lambda_3^{-2} + \lambda_1^{-2}(K^2 + 1), \\ I_3(\mathbf{B}) &= \lambda_1^2\lambda_2^2\lambda_3^2 = 1, \quad I_1(\mathbf{D}) = 0. \end{aligned} \tag{18}$$

In view of the last two relations, the additional simple shearing deformation is isochoric, and the incompressibility conditions (2) are satisfied.

With the aid of (15), (16) and (17), the nonzero components of the Cauchy stress tensor are obtained from (5). We find

$$T_{11} = -p+\beta_1(\lambda_1^2+\lambda_2^2K^2)+\beta_{-1}\lambda_1^{-2}, \tag{19}$$

$$T_{22} = -p+\beta_1\lambda_2^2+\beta_{-1}(\lambda_2^{-2}+K^2\lambda_1^{-2}), \tag{20}$$

$$T_{33} = -p+\beta_1\lambda_3^2+\beta_{-1}\lambda_3^{-2}, \tag{21}$$

$$T_{12} = T_{21} = K\gamma+\eta\dot{K}, \tag{22}$$

wherein the shear response function is defined by

$$\gamma = \lambda_2^2(\beta_1-\lambda_3^2\beta_{-1}). \tag{23}$$

One must bear in mind also the constraint $(18)_3$. It is now clear that when $K \equiv 0$, (19)–(22) yield the uniform equilibrium stress distribution in the homogeneous state mentioned at the start.

Elimination of p between pairs of the normal stress components leads to three relations for the normal stress differences. One of these is related to the shear stress and the viscosity:

$$T_{11}-T_{22} = \left(\frac{\lambda_1^2+\lambda_2^2(K^2-1)}{K\lambda_2^2}\right)(T_{12}-\eta\dot{K}). \tag{24}$$

Thus, when $\eta = 0$ and K is constant, we obtain from (24) the universal relation for nonlinear elastic solids, a result first reported by Wineman and Gandhi (1984) and discussed further by Rajagopal and Wineman (1987), and by Beatty (1989b). However, the same result holds also when $\eta \neq 0$; this happens when the load attains its ultimate equilibrium configuration of shear for which $\dot{K} = 0$, as discussed later on.

Since the response functions are functions of the principal invariants of $\mathbf{B}$ in (18), we have $\beta_\Gamma = \beta_\Gamma(\lambda_1^2, \lambda_2^2, K^2(t))$, $\Gamma = 1, -1$. Thus, with $p = p(t)$, it follows from (19)–(22) that $\operatorname{div}\mathbf{T}(\mathbf{x}, t) = \mathbf{0}$ for all t, and hence the simple shearing deformation superimposed on a homogeneous, triaxial deformation is a controllable, quasi-static deformation. The time varying surface tractions needed to control the shearing motion can now be found; but we shall find no need for them here.

The relation (22) is valid for all incompressible viscoelastic materials in the class (5). It is clear that the shear stress, which is furnished by the load interface, is a function of both the amount of shearing K and the rate of shearing $\dot{K}$. As a result, this leads naturally to discussion of the familiar creep and recovery effects.

4. VISCOELASTIC CREEP AND RECOVERY PHENOMENA IN SIMPLE SHEAR

The creep effect is characterized by growth of the shear deflection $K(t)$ under a constant shearing stress $T_{12} = \tau_{12}$, say. We expect, of course, that if the load is released from the homogeneous state, the shear will increase asymptotically to an ultimate equilibrium state defined by $\dot{K}(t) \to 0$ and $K(t) \to K_s$ as $t \to \infty$. Therefore, the ultimate shear deflection K_s is determined by the constant stress τ_{12} in accordance with (22):

$$\tau_{12} = K_s\gamma(\lambda_1^2, \lambda_2^2, K_s^2). \tag{25}$$

The other ultimate stress components are provided by (19)–(21). Hence, the ultimate equilibrium state of the shear block depends on only the elastic part of the material response evaluated at $K = K_s$; and this is uniquely determined by (25) independently of the viscosity η.

The governing equation for the quasi-static shearing motion is provided by (22) and hence for the creep effect we have

$$-\eta\frac{\mathrm{d}K}{\mathrm{d}t} = K\gamma(\lambda_1^2,\lambda_2^2,K^2(t)) - T_{12} \equiv f(K). \tag{26}$$

Here T_{12} is an arbitrary constant shear stress. Integration of (26) from the homogeneous state where $K = 0$ yields

$$\int_0^K \frac{\mathrm{d}K}{f(K)} = -\frac{t}{\eta}. \tag{27}$$

If an additional shearing load is applied at some intermediate stage so that the constant total shear stress is τ_{12}, the creep continues from this intermediate state with different initial data, but the effect is essentially the same and the ultimate shear K_s is determined by (25).

We now turn to the recovery phenomenon. This is a decay process characterized by a decreasing shear deflection $K(t)$ from an arbitrary initial shear K_0 at which the load is suddenly removed or perhaps suddenly reduced to a lesser value. In particular, if the motion begins from the ultimate state determined by (25) and the load is reduced suddenly to zero so that $T_{12} = 0$, (22) provides the governing equation for recovery:

$$-\eta\frac{\mathrm{d}K}{\mathrm{d}t} = K\gamma(\lambda_1^2,\lambda_2^2,K^2(t)) \equiv g(K). \tag{28}$$

Integration of (28) from the ultimate equilibrium shear $K(0) = K_s$ yields

$$\int_{K_s}^K \frac{\mathrm{d}K}{g(K)} = -\frac{t}{\eta}. \tag{29}$$

If the empirical inequalities (see Truesdell and Noll, 1965; Wang and Truesdell, 1973; Beatty, 1987)

$$\beta_1 > 0, \quad \beta_{-1} \leqslant 0 \tag{30}$$

hold for all deformations of an incompressible, viscoelastic material , it follows from (23) that $\gamma > 0$ holds for all isochoric deformations. Henceforward, we shall assume this holds. Moreover, when the shear response function (23) is known, (27) and (29) may be integrated to determine $K(t)$ in the creep or the recovery process. To go further, therefore, we are forced to consider particular cases. The Mooney–Rivlin model provides a simple example.

4.1. *Creep and recovery of a Mooney–Rivlin material in shear*

A viscoelastic Mooney–Rivlin material is characterized by constant response functions (10). Therefore, the shear response function (23) also is constant. In terms of the usual shear modulus G and the Mooney–Rivlin parameter α in (10), specifically,

$$\gamma = \frac{G\lambda_2^2}{1+\alpha}(1+\alpha\lambda_3^2). \tag{31}$$

When the primary state of the shear blocks is their natural state, all $\lambda_k = 1$ and (31) shows that $\gamma = G$. Hence, the shear response function for a Mooney–Rivlin material from its natural state is independent of α. Suppose, on the other hand, that the triaxial strain is an isochoric uniaxial deformation with stretch λ, so that

$$\lambda_1 = \lambda_3 = \lambda_s^{-1/2}, \quad \lambda_2 = \lambda_s. \tag{32}$$

Then (31) becomes

$$\gamma = \frac{G\lambda_s^2}{1+\alpha}\left(1+\frac{\alpha}{\lambda_s}\right). \tag{33}$$

In this case, the shear response now depends on α; it increases with the uniaxial stretch λ_s in tension and decreases with the stretch in compression. The relation (33) will be useful in discussion of results presented later.

For the neo-Hookean model, $\alpha = 0$ and (31) yields

$$\gamma = G\lambda_2^2. \tag{34}$$

Therefore, the shear response of a neo-Hookean material varies only with the square of the longitudinal stretch λ_2 in the homogeneous state. It has the same form regardless of how the initial deformation is produced. In the usual simple extension with stretch $\lambda_2 = \lambda_s$, $\gamma = G\lambda_s^2$, as seen in (33).

In any event, for a Mooney–Rivlin material, the functions $f(K)$ and $g(K)$ in (26) and (28) are linear in K. Recalling (25), we thus easily obtain from (27) the formula for creep of a viscoelastic Mooney–Rivlin material in shear:

$$K(t) = K_s(1-e^{-\gamma t/\eta}). \tag{35}$$

Integration of (29) delivers the equation for recovery of a viscoelastic Mooney–Rivlin material in shear:

$$K(t) = K_s e^{-\gamma t/\eta}. \tag{36}$$

In either case, it is seen that $0 \leqslant K/K_s \leqslant 1$ for all $K(t)$. The results (35) and (36) will be discussed in turn. We begin with the creep relation (35).

Since $\gamma > 0$, when the load is released from its initial state (35) shows that the shear deflection asymptotically approaches its ultimate equilibrium value $K = K_s$ as $t \to \infty$, which was anticipated earlier. Theoretically, it takes an infinitely long time to complete the creep process. On the practical side, however, we need some measure of how fast the shear deflection creeps to the final equilibrium state. The retardation time t_r defined by

$$t_r \equiv \frac{\eta}{\gamma} \tag{37}$$

provides a measure of this property. By (35), the ratio K/K_s at $t = t_r$ determines the constant retardation ratio

$$\frac{K(t_r)}{K_s} = 1-e^{-1} \doteq 0.632. \tag{38}$$

This is a universal constant for all Mooney–Rivlin materials. Therefore, the retardation time t_r is the time during which the amount of shear attains 63.2% of its ultimate value in

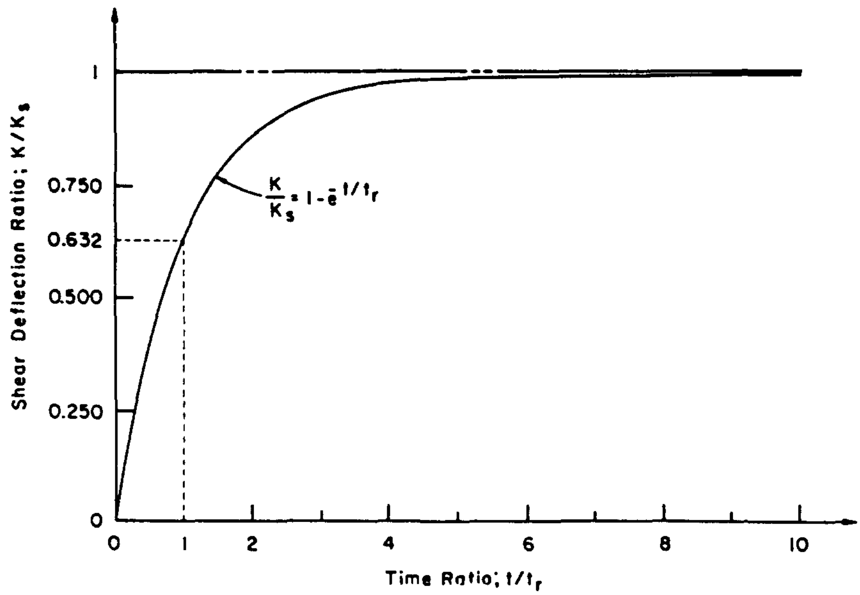

Fig. 2. Creep response of a viscoelastic Mooney–Rivlin material in a simple shear superimposed on a pure homogeneous strain.

an uninterrupted creep process from the homogeneous state. The graph of (35) shown in Fig. 2 is typical of the creep response in shear.

In principle, the values of K_s and t_r may be found by experiment. We suppose that $K(t)$ is measured in a shear experiment in which the value of K_s is obtained as the ultimate amount of shear. Then the ratio $\gamma/\eta = 1/t_r$ may be read from the slope of the semilog plot of test data for $y(t) \equiv \log(1-K/K_s) = -t/t_r$.

It is evident in (37) that the retardation time varies inversely with the shear response γ. In a primary uniaxial deformation of the shear blocks, for example, (33) shows that γ increases in a simple tension with stretch $\lambda_s > 1$ and decreases in a compression with $\lambda_s < 1$. We suppose, of course, that the latter is a stable equilibrium configuration. Therefore, as shown in Fig. 3, the retardation time may be decreased by prestretching the blocks in

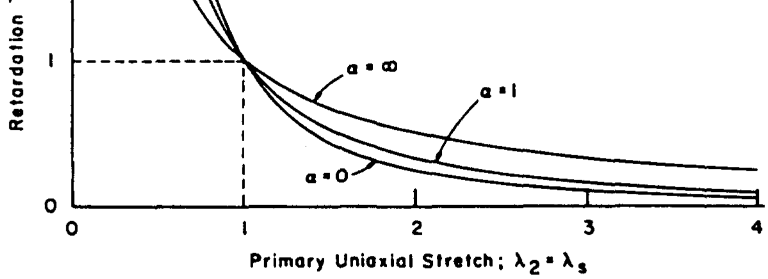

Fig. 3. Retardation time ratio t_r/t_1 as a function of the stretch λ, in a primary simple extension of the shear blocks for three values of the Mooney–Rivlin parameter α. The graph for $\alpha = 0$, however, is valid for an arbitrary homogeneous deformation of a neo-Hookean material.

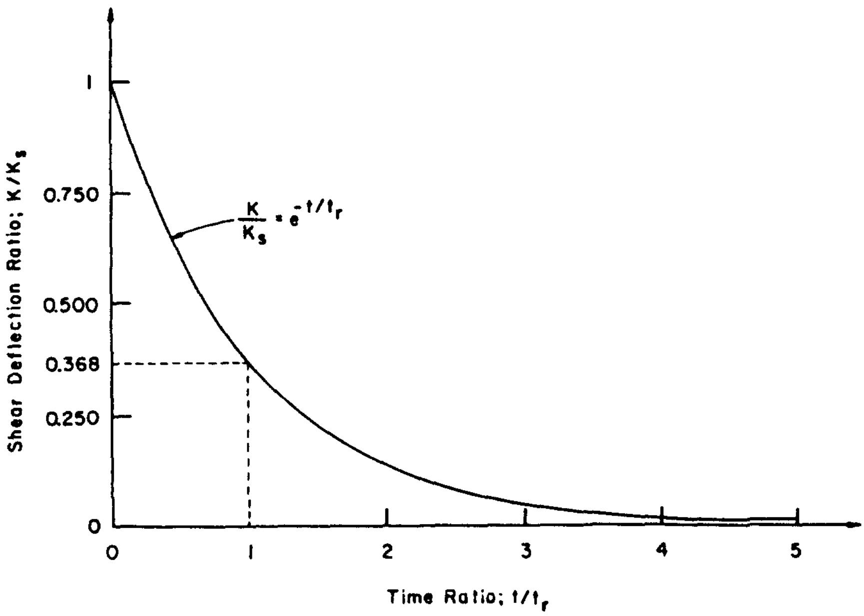

Fig. 4. Recovery response of a viscoelastic Mooney–Rivlin material in a simple shear superimposed on a pure homogeneous deformation.

tension and increased by their compression; that is, creep occurs more quickly when the blocks are prestretched, more slowly when precompressed. In the unstretched case with $\lambda_s = 1$, the retardation time in creep from the natural state is a constant $t_r = t_1 \equiv \eta/G$ independent of α. In general, the retardation time (37) is a monotone decreasing function of $\lambda_s \in (0, \infty)$ for each α. It grows indefinitely large when $\lambda_s \to 0$ and becomes very small when $\lambda_s \to \infty$, as shown in Fig. 3. The variation of the retardation time curves for a few values of α also are mapped in Fig. 3. The case $\alpha = 0$ describes the neo-Hookean model. One sees, however, that these curves are relatively insensitive to variations of $\alpha \in [0, \infty)$, the usual value being smaller than 1. Precisely, for $\alpha = \infty$, $t_r/t_1 = \lambda_s^{-1}$, whereas for $\alpha = 0$, $t_r/t_1 = \lambda_s^{-2}$; and for $\alpha = 1$, $t_r/t_1 = 2/[\lambda_s(\lambda_s + 1)]$. Moreover, for neo-Hookean shear mounts subjected to an arbitrary homogeneous strain, $t_r/t_1 = \lambda_2^{-2}$, follows from (34) and (37). Thus, the curve for $\alpha = 0$ in Fig. 3 is valid more generally for an arbitrary axial stretch λ_2 in the predeformed state of a viscoelastic neo-Hookean material. This important practical example thus shows that an initial longitudinal stretch λ_s plays a more significant role than the material constant α in the variation of the retardation time and in the physical characterization of their effects in creep.

The graph of (36) in Fig. 4 shows the response typical of recovery in shear. The recovery starts from the ultimate shear K_s when the shear stress τ_{12} is suddenly removed. In this case, the amount of shear $K(t)$ approaches the unsheared, homogeneous state asymptotically as $t \to \infty$. It can be shown that the retardation time (37) is the time during which the shear recovers by 63.2% from its ultimate value in an uninterrupted recovery process. Said differently, t_r is the time for the shear to recover to within 36.8% of its null value. The recovery ratio $K(t_r)/K_s = e^{-1} \doteq 0.368$ given by (36) is a universal constant for all Mooney–Rivlin materials. For recovery, the dependence of the retardation time on λ_s and α is exactly the same as shown in Fig. 3 for creep. Hence, as before, the retardation time in the recovery process is decreased by extension of the blocks and increased by their contraction, which otherwise we suppose is stable. This means that recovery of the load supported by springs under tension always is faster than recovery when they are compressed.

This completes our study of the creep and recovery properties of a viscoelastic Mooney–Rivlin material in a simple shear superimposed on a primary homogeneous deformation. We next consider viscoelastic effects in the vibration problem of the shearing oscillator shown in Fig. 1.

5. FINITE AMPLITUDE VIBRATIONS OF THE SHEARING OSCILLATOR

In this section, the finite amplitude, damped, free vibration of a load supported symmetrically betweeen identical viscoelastic shear mountings subjected initially to the same homogeneous deformation will be investigated. The equation of motion is formulated for an incompressible, isotropic viscoelastic material of general type (5). Afterwards, the problem is solved exactly for a viscoelastic Mooney–Rivlin material. Some results for nonlinearly elastic solids are discussed at the end.

To begin, we note that the engineering stress $\mathbf{S}$ on an incompressible material is determined by $\mathbf{S} = \mathbf{T}\mathbf{F}^{-T}$ (Beatty, 1987). Hence, with the use of (14), the engineering shear stress S_{12} at the shear block interface with the load is given by $S_{12} = T_{12}/\lambda_2$, in which we recall (22). Also, use of (13) gives the acceleration $\ddot{x} = \lambda_2 L\ddot{K}$ of the interface identified by $Y = L$ in the natural state where its area is A. As usual, the inertia of the shear blocks themselves will be ignored. We shall assume that the quasi-static shear stress at the bonded load interface approximates the shear stress in the dynamical problem. Of course, in view of the symmetry, the normal tractions exerted on M by the shear blocks are balanced at all times. Hence, with reference to the system in Fig. 1, the general nonlinear equation of motion for a load M supported symmetrically by incompressible, isotropic viscoelastic shear mountings is given by

$$\ddot{K}+\frac{2A\eta}{ML\lambda_2^2}\dot{K}+\frac{2A}{ML\lambda_2^2}\gamma(\lambda_1^2,\lambda_2^2,K^2)K=\frac{g^*}{L\lambda_2},\tag{39}$$

wherein we recall (23) and write $g^* \equiv g\sin\theta$ for the gravitational acceleration component. In a motion of M on a smooth horizontal surface, we set $g^* = 0$ in (39). In either case, the nature of the differential equation (39) will depend on the form of the shear response function. We thus turn to a particular example.

5.1. *Damped vibrations on viscoelastic Mooney–Rivlin springs*

The shear response function for a Mooney–Rivlin material is a constant given in (31). Hence, the differential equation (39) for the finite amplitude motion of a shearing oscillator simplifies to the familiar equation for a linear damped oscillator:

$$\ddot{K}+\frac{2A\eta}{ML\lambda_2^2}\dot{K}+\frac{2A\gamma}{ML\lambda_2^2}K=\frac{g^*}{L\lambda_2}.\tag{40}$$

It is seen that the ultimate equilibrium position K_s obtained from (40) is

$$K_s = p^2\frac{ML\lambda_2^2}{2A\gamma}\quad\text{with}\quad p^2\equiv\frac{g^*}{L\lambda_2}.\tag{41}$$

For our Mooney–Rivlin model, this equilibrium equation is equivalent to (25); it identifies in specific terms the ultimate amount of shear in the quasi-static creep and recovery solutions (35) and (36). Using the first relation in (41) to eliminate the mass from (40), we obtain the equation for the damped, free vibrational motion of our Mooney–Rivlin shearing oscillator in the form

$$\ddot{\kappa}+2\nu\dot{\kappa}+\omega^2\kappa = 0,\tag{42}$$

wherein

$$\kappa(t)\equiv K(t)-K_s,\quad 2\nu\equiv p^2\frac{t_r}{K_s}=\omega^2 t_r,\quad \omega^2\equiv\frac{p^2}{K_s}=\frac{p_0^2}{\lambda_2 K_s},\tag{43}$$

and t_r is the retardation time (37). From $(41)_2$, we identify $p_0^2 \equiv g^*/L$. We shall suppose that $K_s \neq 0$, i.e. $g^* \neq 0$. The case when $K_s = 0$ is treated separately later on.

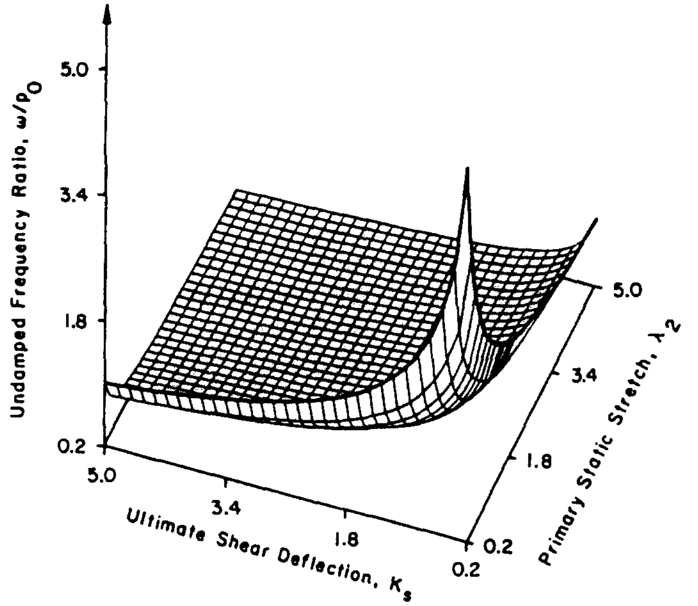

Fig. 5. Universal undamped, free vibrational frequency ratio as a function of λ_2 and K_s for viscoelastic Mooney–Rivlin shear mountings subjected initially to an isochoric, homogeneous deformation.

The nature of the solution of (42) will depend on the undamped, free vibrational frequency ω and the damping coefficient v defined in (43). Both are functions of the primary deformation of the shear blocks and the ultimate shear deflection. So before we discuss the solution of (42), the damping and frequency terms in (43) will be examined.

5.1.1. *Frequency and damping ratios for Mooney–Rivlin springs.* The last expression in (43) reveals that the frequency ratio ω/p is a universal relation for all viscoelastic Mooney–Rivlin shear mountings. For a fixed homogeneous state, the ratio is determined strictly by the amount of the static shear deflection. Hence, test data for the undamped, free vibrational frequency of a simple shearing oscillator do not distinguish one Mooney–Rivlin shear spring from any other having the same shear deflection. The frequency ratio ω/p_0 is a monotone decreasing function of both λ_2 and K_s. Hence, an increase in either static deformation of the springs lowers the frequency; a decrease of either will increase it; and for small K_s, ω may be very large. This universal frequency response is illustrated in the three-dimensional surface plot in Fig. 5. Although the diagram shows a wide range for K_s, since $K = \tan\psi$, where ψ is the angle of shear shown in Fig. 1, it is clear that in practical cases $K_s < 3/5$ ($\psi < 30^\circ$), say. Of course, the possibility of larger values is not excluded.

Returning to the second relation in (43), we find by use of (31) and (37) the damping ratio for our viscoelastic Mooney–Rivlin oscillator:

$$\frac{v}{v_0} = \frac{1+\alpha}{\lambda_2^3 K_s(1+\alpha\lambda_3^2)} \quad \text{with} \quad 2v_0 \equiv \frac{\eta p_0^2}{G}. \tag{44}$$

Thus, for a given homogeneous state, the damping ratio decreases as K_s increases under the load. It is seen that in every shearing motion from the natural state with $\lambda_k = 1$, (44) yields the universal damping ratio

$$\frac{v}{v_0} = \frac{1}{K_s}. \tag{45}$$

Further, when the primary deformation is induced by uniaxial loading with stretch $\lambda_2 = \lambda_s$, we may recall (32) and (33) and thus note that the damping ratio (44) is decreased by

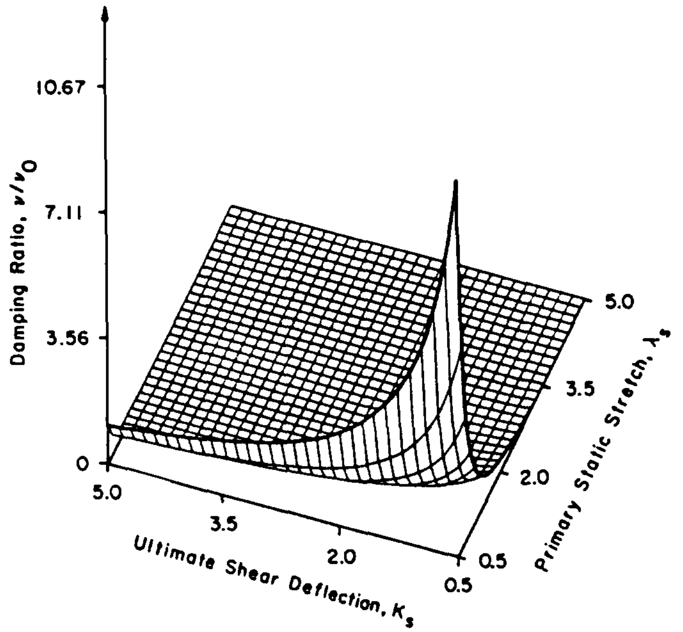

Fig. 6. Damping ratio as a function of λ_s and K_s for viscoelastic Mooney–Rivlin shear mountings for the case $\alpha = 1$ and subjected initially to an isochoric, simple extension.

pretension and increased by precompression. Also, in this case, the dependence on α in $(43)_2$ is determined by t_r in accordance with Fig. 3; hence, we may expect no great variation of damping with α. In general, for a fixed elongation, the damping ratio will increase with α; for a fixed compression, it will decrease when α is increased. The damping ratio provided by (44) under a primary uniaxial loading is shown in Fig. 6 for $\alpha = 1$.

Finally, we see from (44) that with $\alpha = 0$ the damping ratio for neo-Hookean springs is given by the universal relation

$$\frac{\nu}{\nu_0} = \frac{1}{\lambda_2^3 K_s} \tag{46}$$

valid for any primary isochoric, homogeneous deformation of the shear blocks. Thus, the damping decreases significantly with increasing extensional stretch, and grows greater with increasing compression. We next consider the appropriate solution of (42).

5.1.2. *Description of the damped vibrational motion.* The general solution of (42) is well-known. Since the damping coefficient ν and the undamped, free vibrational frequency ω in (43) are positive, the solution of (42) is asymptotically stable at $K = K_s$ for all physically possible initial data. If $\nu > \omega$, that is, with (37) and $(43)_2$, if

$$\eta > 2\frac{\gamma}{\omega}, \tag{47}$$

the motion is heavily damped. The relation (47) simply indicates that a heavily damped motion is possible only when the material viscosity η is sufficiently large, which is intuitively natural. When the inequality in (47) is replaced by equality, the system is critically damped and we write $\eta = \eta^*$ and $\nu = \nu^*$, where

$$\nu^* = \omega, \quad \eta^* = 2\frac{\gamma}{\omega}. \tag{48}$$

The solution of (42) in the heavily damped case is given by

$$\kappa(t) = K(t) - K_s = e^{-\nu t}(Ae^{\zeta t} + Be^{-\zeta t}), \tag{49}$$

in which

$$\zeta = \sqrt{\nu^2 - \omega^2}. \tag{50}$$

And for the critically damped case, we have

$$\kappa(t) = K(t) - K_s = (A + Bt)e^{-\nu^* t}. \tag{51}$$

The constants A and B in (49) and (51) are determined by the initial data, as usual.

Whether either of the over-damped solutions (49) or (51) may hold will depend upon the extent of the primary deformation of the shear mounts, in addition to the nature of the material constants. If the homogeneous state is an isochoric, simple extension so that (33) applies, for example, we see from (48) and the last relation in (43) that upon increasing the stretch λ_s by tension, or by increasing the static shear deflection K_s, the critical number η^* in (48) will increase, (47) may then fail to hold, and hence a heavily damped motion is less likely to occur. On the other hand, damping of the system may be effectively increased by sufficient compressional deformation to reduce the shear response in (33) and hence also the critical number in (48). In this case, (47) might hold.

We shall now consider the oscillatory motion of the load with light damping so that the ν is small compared with ω. Then the solution to (42) may be written as

$$\kappa = K - K_s = Q_0 e^{-\nu t} \sin[\omega_d t + \phi_0], \tag{52}$$

wherein the damped circular frequency ω_d is defined by

$$\omega_d = \sqrt{\omega^2 - \nu^2}, \tag{53}$$

and Q_0 and ϕ_0 are real constants determined by assigned initial conditions. We see from (52) that if the load passes the position $\kappa = 0$ in a given direction at time t_0, it will pass $\kappa = 0$ in the same direction at time $t_0 + 2\pi/\omega_d$. Although the motion is not periodic, the time τ defined by

$$\tau \equiv \frac{2\pi}{\omega_d} \tag{54}$$

usually is called the period of the lightly damped motion. Thus, the lightly damped, finite amplitude motion of a load M supported by viscoelastic Mooney–Rivlin shear springs is a harmonic, but nonperiodic oscillatory motion with damped circular frequency (53) and having an exponentially decreasing amplitude ending at the static shear deflection K_s, the center of the oscillations.

The rate of the amplitude decay usually is measured by the logarithmic decrement δ which is given by

$$\delta = \log\frac{\kappa_j}{\kappa_{j+1}} = \frac{2\pi\nu}{\omega_d} = \frac{2\pi\nu}{\sqrt{\omega^2 - \nu^2}}, \tag{55}$$

where $\kappa_j(t)$ and $\kappa_{j+1}(t+\tau)$ are two successive amplitudes of the motion (52). When ν is small compared with ω, $\delta \approx 2\pi\nu/\omega$. Hence, by (43), (31), and (37), we have approximately

$$\delta \approx \pi\omega t_r = \frac{\pi\omega\eta}{\gamma} = \frac{\pi p_0\eta(1+\alpha)}{G\lambda_2^{5/2}(1+\alpha\lambda_3^2)\sqrt{K_s}}. \tag{56}$$

We see that δ varies inversely with the deformation of the shear blocks. If δ is measured in the usual manner from the amplitudes of the damped response curve in a free vibration of the system, it is seen that (56), or similar relations of this kind, provide a means to determine the viscosity coefficient η, or the ratio η/γ, for example, when the other data may be known. We note that δ also is a measure of the amount of energy dissipated in each cycle of the decaying motion; however, we leave discussion of this matter for another time.

We recall that the solution for the viscoelastic neo-Hookean model is the special case for which $\alpha = 0$ everywhere above. In particular, for the neo-Hookean material, we have from (56),

$$\delta = \frac{\pi p_0\eta}{G\lambda_2^{5/2}\sqrt{K_s}}. \tag{57}$$

Thus, by increasing λ_2 in an arbitrary, primary homogeneous deformation of the springs, or by increasing K_s, the lightly damped motion of the neo-Hookean oscillator is made more lively, the amplitude decays more slowly. The amplitude decays more rapidly, when the blocks are initially compressed.

We notice also that if the shear blocks of a Mooney–Rivlin oscillator are assembled without primary deformation, (31) yields $\gamma = G$. Therefore, in this case, none of the foregoing relations will depend on the Mooney–Rivlin parameter α. Hence, the viscoelastic vibrational effects on the motion of a load supported by initially undeformed Mooney–Rivlin springs with moduli G and η cannot be distinguished from those for neo-Hookean springs of the same design and having the same two material constants. In (56), for example, the logarithmic decrement for lightly damped motions of a load on initially undeformed Mooney–Rivlin springs is given by

$$\delta = \frac{\pi p_0\eta}{G\sqrt{K_s}}, \tag{58}$$

which is exactly the same as the general rule (57) when $\lambda_2 = 1$.

Moreover, for a primary uniaxial stress, we have seen earlier in Fig. 3 that if $\lambda_s \neq 1$, an increase in α at a fixed stretch results in an increase in the retardation time in a primary simple tension, and a decrease in compression. Hence, δ in (56) increases with α in a given simple tension. For a fixed elongation of the springs, therefore, an increase in α leads to an effective increase in the damping, as described earlier. An increase in α when the springs are under a given initial compression, as seen from Fig. 3, has the opposite, possibly greater, effect in reducing the effective damping. We next examine the case when $K_s = 0$.

5.1.3. *Motion on a smooth horizontal surface*. The motion of the load M on a smooth horizontal surface is described by (40) in which $g^* = 0$; and hence the static equilibrium shear deflection is $K_s = 0$. In this case, the motion is governed by (42) in which $\kappa = K$ and the damping coefficient v and undamped frequency ω are defined by

$$2v = \frac{2A\eta}{ML\lambda_2^2}, \quad \omega^2 = \frac{2A\gamma(\lambda_1^2, \lambda_2^2)}{ML\lambda_2^2}. \tag{59}$$

The trivial equilibrium condition cannot be used to remove the load from (59); but we still have $2v = \omega^2 t_r$. Since v and ω are positive, the motion is asymptotically stable at $K_s = 0$ for all physically possible initial data. If (48) and (49) may hold, the motion is over-damped and the solution with $K_s = 0$ is given by (50) and (51), respectively. Otherwise, the motion is lightly damped and the solution is given by (52) with $K_s = 0$. The effect of the primary homogeneous deformation on the damping and the logarithmic decrement is similar to the previous case. Hence, in each instance, the horizontal motion begins from an assigned initial state and ultimately comes to rest in the primary equilibrium state where $K_s = 0$. How it does this is determined, not only by the material constants, but also by the nature of the initial static deformation of the springs.

This concludes our study of the free vibrations of a load supported symmetrically by viscoelastic shear mountings. Some further results for shear and other deformations of isotropic viscoelastic materials in the class (5) will be presented in future papers. We end our work here with a brief mention of the general solution possible for perfectly elastic shear springs.

5.2. *Free vibrations on nonlinearly elastic shear mountings*

For the undamped, perfectly elastic case when $\eta = 0$, (39) may be readily integrated to obtain the energy integral

$$\dot{K} = \pm\left[\dot{K}_0^2 - \frac{2A}{ML\lambda_2^2}\int_{K_0^2}^{K^2} \gamma(\lambda_1^2, \lambda_2^2, K^2)\,\mathrm{d}K^2 + \frac{2g^*}{L\lambda_2}(K-K_0)\right]^{1/2}, \tag{60}$$

where $K(0) \equiv K_0$ and $\dot{K}(0) \equiv \dot{K}_0$. A second integration yields the travel time

$$t = \int_{K_0}^{K} \frac{\mathrm{d}K}{\dot{K}(K)}. \tag{61}$$

These equations have been applied by Beatty (1984, 1988) to study shearing oscillations of the load from the natural state of isotropic elastic shear mountings. However, the current formulation admits an arbitrary homogeneous deformation of the shear blocks.

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